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A SUPERPOSITION PRINCIPLE IN OPTIMAL PLASTIC DESIGN FOR ALTERNATIVE LOADS†

J. C. NAGTEGAAL

Division of Engineering, Brown University, Providence, R.I., U.S.A.

Abstract—The paper is concerned with the optimal plastic design of sandwich beams, frames and trusses for alternative loading conditions. Upper and lower bounds for the optimal weight of a beam are derived, for single as well as for alternative loading conditions. These bounding theorems are used to establish a superposition principle. If no explicit bounds on the cross-sectional areas are prescribed, the optimal design for alternative loading conditions $\mathbf{P}_1$ and $\mathbf{P}_2$ can be obtained by superposition of the optimal designs for the single loading conditions $\frac{1}{2}(\mathbf{P}_1+\mathbf{P}_2)$ and $\frac{1}{2}(\mathbf{P}_1-\mathbf{P}_2)$. If the cross-sections are to have at least given non-zero values, the principle furnishes upper and lower bounds to the optimal weight.

The principle is illustrated by a simple example.

1. INTRODUCTION

IN A RECENT paper [1], it has been shown that the optimal limit design of trusses under alternative loading conditions can be obtained by the superposition of the optimal designs for two single loading conditions. Since then, the attention of the authors has been drawn to a set of lecture notes [2], in which a similar result was obtained from a linear programming formulation. In the present paper, the superposition principle is established in a more general form that applies to sandwich beams and frames as well as trusses. When the minimum yield force of a bar in a truss or the minimum plastic moment of a beam is prescribed, the superposition principle yields upper and lower bounds on the structural weight. When the prescribed cross-sectional values are zero, the superposition principle furnishes the exact solution. In the derivation of these results, bounding theorems on the optimal weight are used that are comparable to the general duality statement given by Rozvany and Adidam [3].

For brevity, the following discussion is restricted to sandwich beams; the results, however, apply as well to sandwich frames and to trusses.

2. FORMULATION OF PROBLEM, OPTIMALITY CONDITION

For a sandwich beam with rectangular core cross-section of constant width b and constant height $2h$ and identical cover plates of varying thickness $t(x) \ll h$, the volume of the cover plates per unit length is proportional to the plastic moment $M_p(x)$. Since the weight of the core is fixed, minimization of the weight of the beam is equivalent to minimization

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of the weight (or volume) of the cover plates, or minimization of the integral

$$W = \int M_p \, \mathrm{d}x. \tag{2.1}$$

Consider a sandwich beam of this kind that is to have a given load factor λ for plastic collapse when subject to a given load distribution $p(x)$. It will be assumed that the plastic moment is bounded from below by

$$M_p(x) \geq M_0(x), \tag{2.2}$$

where $M_0(x)$ is given. If no explicit bound is prescribed, there still is the requirement that $M_p(x)$ must be nonnegative ($M_0(x) = 0$).

As was pointed out by Heyman [4] for $M_0(x) = 0$ and by Prager [5] for $M_0(x) > 0$, a design is optimal if it admits a collapse mode under the given loads such that the curvature rate $\kappa(x)$ satisfies

$$|\kappa(x)| \begin{cases} = 1 & \text{if} \quad M_p(x) > M_0(x), \\ \leq 1 & \text{if} \quad M_p(x) = M_0(x). \end{cases} \tag{2.3}$$

Note that this condition is *necessary* and *sufficient* for *global* optimality.

If the beam is subjected to the alternative loadings $p'(x)$ and $p''(x)$ for each of which the load factor is not to be smaller than λ, a similar optimality condition can be derived. From different points of view, Chan [6] and Prager [7] have shown that a design is optimal if it admits collapse modes for each loading with curvature rates $\kappa'(x)$ and $\kappa''(x)$ satisfying

$$|\kappa'(x)| + |\kappa''(x)| \begin{cases} = 1 & \text{if} \quad M_p(x) > M_0(x), \\ \leq 1 & \text{if} \quad M_p(x) = M_0(x). \end{cases} \tag{2.4}$$

Note, that if one of the loads is not relevant, the corresponding curvature rate vanishes, and the optimality condition (2.4) reduces to (2.3).

3. LOWER BOUND THEOREMS

The lower bound theorem for single loading has the following form. If $v^*(x)$ is a kinematically admissible velocity field with positive rate of dissipation $\int pv^* \, \mathrm{d}x$ that has been scaled in such a manner that the rate of curvature $\kappa^*(x)$ satisfies

$$|\kappa^*(x)| \leq 1, \tag{3.1}$$

then

$$W_{\mathrm{opt}} \geq \lambda \int pv^* \, \mathrm{d}x, \tag{3.2}$$

where λ is the prescribed load factor for the given loads $p(x)$†.

† As was pointed out by one of the referees, one can obtain an improved lower bound‡ if a term $\int M_0(1-|\kappa^*|) \, dx$ is added to the term $\lambda \int pv^* \, dx$ in (3.2). The bound thus obtained can, however, not be used to establish the superposition principle.

‡ D. E. CHARRETT, Ph.D. Thesis, Monash University (1970).

Proof

Let $M_p(x)$ be the optimal design and $\kappa(x)$ be a collapse mode that satisfies (2.3). It then follows from the kinematic theorem of limit analysis (see, for instance, [8]) that

$$\lambda = \int |\kappa| M_p \, \mathrm{d}x \Big/ \int pv \, \mathrm{d}x \leq \int |\kappa^*| M_p \, \mathrm{d}x \Big/ \int pv^* \, \mathrm{d}x. \tag{3.3}$$

Use of (3.1) in (3.3) immediately furnishes (3.2).

It should be noted that use of the optimality condition (2.3) with $M_0(x) \equiv 0$ in the first part of (3.3) yields

$$W_{\mathrm{opt}} = \lambda \int pv \, \mathrm{d}x. \tag{3.4}$$

Accordingly, if $M_0(x) \equiv 0$, then W_{opt} is the largest value of the integral on the right of (3.2) for all kinematically admissible velocity fields that are scaled according to (3.1). If, on the other hand, $M_0(x) \not\equiv 0$, use of (2.3) in (3.3) only yields

$$W_{\mathrm{opt}} \geq \lambda \int pv \, \mathrm{d}x. \tag{3.5}$$

The lower bound theorem for alternative loading conditions resembles the theorem just discussed. If $v^{*\prime}(x)$ and $v^{*\prime\prime}(x)$ are kinematically admissible velocity fields which have been scaled in such a manner that the rates of curvature $\kappa^{*\prime}(x)$ and $\kappa^{*\prime\prime}(x)$ satisfy

$$|\kappa^{*\prime}(x)| + |\kappa^{*\prime\prime}(x)| \leq 1, \tag{3.6}$$

then

$$W_{\mathrm{opt}} \geq \lambda \int (p' v^{*\prime} + p'' v^{*\prime\prime}) \, \mathrm{d}x. \tag{3.7}$$

Proof

Let $M_p(x)$ be the optimal design and $\kappa'(x)$ and $\kappa''(x)$ be collapse modes that satisfy (2.4). The argument that led to (3.3) now yields:

$$\int |\kappa^{*\prime}| M_p \, \mathrm{d}x \geq \lambda \int p' v^{*\prime} \, \mathrm{d}x, \tag{3.8a}$$

and

$$\int |\kappa^{*\prime\prime}| M_p \, \mathrm{d}x \geq \lambda \int p'' v^{*\prime\prime} \, \mathrm{d}x. \tag{3.8b}$$

Summation of (3.8a) and (3.8b) and use of (3.6) then furnishes (3.7). Note that (3.7) has the same extremum properties as (3.2).

4. UPPER BOUND THEOREMS

The upper bound theorem for single loading follows directly from the static theorem of limit analysis [8]. If $M^*(x)$ is a statically admissible bending moment distribution,

$$W_{\mathrm{opt}} \leq \int \max\{\lambda |M^*(x)|, M_0(x)\} \, \mathrm{d}x. \tag{4.1}$$

Proof

By the static theorem of limit analysis, the design

$$M_p^*(x) = \max\{\lambda|M^*(x)|, M_0(x)\} \tag{4.2}$$

has a load factor not smaller than λ; since this design also satisfies (2.2), it is *feasible*, and this establishes (4.1).

The proof of the upper bound theorem for alternative loading conditions follows the same lines and will be omitted for brevity. If $M^{*\prime}(x)$ and $M^{*\prime\prime}(x)$ are statically admissible bending moments for the loads $p'(x)$ and $p''(x)$, then

$$W_{\text{opt}} \leq \int \max\{\lambda|M^{*\prime}(x)|, \lambda|M^{*\prime\prime}(x)|, M_0(x)\}\,\mathrm{d}x. \tag{4.3}$$

It should be noted that if $M^*(x)$ in (4.1) is the actual bending moment of the optimal design in a collapse mode satisfying (2.3), then

$$W_{\text{opt}} = \int M_p\,\mathrm{d}x = \int \max\{\lambda|M(x)|, M_0(x)\}\,\mathrm{d}x. \tag{4.4}$$

Accordingly, W_{opt} is the smallest value of the integral on the right of (4.1) for all statically admissible bending moments $M^*(x)$. A similar remark applies to (4.3).

5. SUPERPOSITION OF OPTIMAL DESIGNS

As has been shown by Rozvany and Adidam [3], the theorems discussed in the preceding sections are valuable in the direct application to optimal design problems. As is shown below, these theorems may also be used to establish the validity of a principle of superposition of optimal designs.

Let a suitably supported sandwich beam be subject to the alternative loads $p'(x)$ and $p''(x)$, while the plastic moment $M_p(x)$ has to satisfy (2.2). The minimal weight of the cover plates of the beam under these conditions will be denoted by W_{opt}.

Instead of this alternative loading problem, consider the two single loading problems where beams that are supported in the same manner have to carry the loads

$$p^+(x) = \tfrac{1}{2}(p'(x)+p''(x)), \qquad p^-(x) = \tfrac{1}{2}(p'(x)-p''(x)), \tag{5.1}$$

while the plastic moments are constrained by

$$M_p^+(x) \geq \tfrac{1}{2}M_0(x), \qquad M_p^-(x) \geq \tfrac{1}{2}M_0(x). \tag{5.2}$$

The principle of superposition may then be stated as follows. If the lower bounds W_l^+, W_l^- and the upper bounds W_u^+, W_u^- for the minimum weight of the cover plates in the single loading problems (5.1), (5.2) are obtained by the use of (3.2) and (4.2), the minimum weight W_{opt} for the alternative loadings p', p'' satisfies

$$W_l^+ + W_l^- \leq W_{\text{opt}} \leq W_u^+ + W_u^-. \tag{5.3}$$

Proof

Since W_l^+ and W_l^- were obtained by the use of (3.2), there exist velocity fields $v^+(x)$ and $v^-(x)$ such that

$$|\kappa^+(x)| \leq 1, \qquad |\kappa^-(x)| \leq 1, \tag{5.4}$$

and

$$W_l^+ + W_l^- = \lambda \int (p^+ v^+ + p^- v^-)\,dx. \tag{5.5}$$

Since the velocity fields $v^+(x)$ and $v^-(x)$ are kinematically admissible, the fields

$$v'(x) = \tfrac{1}{2}(v^+(x) + v^-(x)), \qquad v''(x) = \tfrac{1}{2}(v^+(x) - v^-(x)) \tag{5.6}$$

are also kinematically admissible. Now, with use of (5.4),

$$|\kappa'(x)| + |\kappa''(x)| = \tfrac{1}{2}|\kappa^+(x) + \kappa^-(x)| + \tfrac{1}{2}|\kappa^+(x) - \kappa^-(x)| \le 1, \tag{5.7}$$

since the sum of the absolute values is equal to either $\kappa^+(x)$ or $\kappa^-(x)$. Hence, from (3.6), (3.7), (5.1), (5.5) and (5.6) it follows that

$$W_{\text{opt}} \ge \lambda \int (p'v' + p''v'')\,dx = \frac{\lambda}{2} \int [(p^+ + p^-)(v^+ + v^-)$$

$$+ (p^+ - p^-)(v^+ - v^-)]\,dx = \lambda \int (p^+ v^+ + p^- v^-)\,dx = W_l^+ + W_l^-. \tag{5.8}$$

This establishes the first part of (5.3).

The second inequality in (5.3) is established in the following manner. According to (4.1), there exist statically admissible bending moments satisfying

$$W_u^+ + W_u^- = \int [\max\{\lambda|M^+(x)|, \tfrac{1}{2}M_0(x)\} + \max\{\lambda|M^-(x)|, \tfrac{1}{2}M_0(x)\}]\,dx$$

$$\ge \int \max\{\lambda|M^+(x)| + \lambda|M^-(x)|, M_0(x)\}\,dx. \tag{5.9}$$

In view of (5.1) there thus exist statically admissible bending moments $M'(x)$ and $M''(x)$ satisfying

$$\lambda|M^+(x)| + \lambda|M^-(x)| = \frac{\lambda}{2}|M'(x) + M''(x)| + \frac{\lambda}{2}|M'(x) - M''(x)|$$

$$= \max\{\lambda|M'(x)|, \lambda|M''(x)|\}. \tag{5.10}$$

From (5.9), (5.10) and (4.3) it therefore follows that

$$W_u^+ + W_u^- \ge \int \max\{\lambda|M'(x)|, \lambda|M''(x)|, M_0(x)\}\,dx \ge W_{\text{opt}}. \tag{5.11}$$

This establishes the second part of (5.3).

Note, that, for the general case $M_0(x) \ge 0$, the inequalities (5.3) do not lead to extremum principles, and hence it will, in general, not be possible to make the upper and lower bounds coincide. In the important special case $M_0(x) = 0$, however, the inequalities (5.3) provide extremum principles as was pointed out in Section 3. Moreover, if $M_p^+(x)$ and $M_p^-(x)$ are the plastic moments in the optimal designs for $p^+(x)$ and $p^-(x)$, the plastic moments in the optimal design for the alternative loads $p'(x)$ and $p''(x)$ are

$$M_p(x) = M_p^+(x) + M_p^-(x), \tag{5.12}$$

and consequently

$$W_{\text{opt}} = W^{+}_{\text{opt}} + W^{-}_{\text{opt}}. \tag{5.13}$$

These last results are particularly useful in the solution of alternative loading problems. It makes the application of analytical techniques possible (as in the case of Michell trusses [1]), or simplifies the application of such techniques (see example). Larger truss optimization problems are often solved with help of linear programming techniques [9–11]. The computing cost of this kind of solution, however, increases rapidly with the number of variables involved. The present formulation reduces the alternative loading to two smaller problems, and this may cause a considerable reduction in computing cost.

6. EXAMPLE

Consider a sandwich beam of the type described in Section 2, with length $\frac{3}{2}l$ that is built in at $x = 0$, simply supported at $x = l$ and subjected to alternative loads shown in Fig. 1(a–b). With the use of (5.1), this problem is converted into two problems of single loading, shown in Fig. 1(c–d). In order to solve these single loading problems, first consider collapse mechanisms that satisfy (2.3). Since it is not likely that $M_p(x)$ will vanish in any finite region in either problem, $|\kappa(x)| = 1$ anywhere. Possible mechanisms are shown in Fig. 1(e–f); note that in the mechanism shown in Fig. 1(e), compatibility requires

$$\eta_1^2 - \eta_0^2 = \tfrac{1}{2}. \tag{6.1}$$

The next step is to consider statically admissible bending moments, that can be expressed in terms of the reactions x^+ and x^- at the simple support. The bending moment diagrams

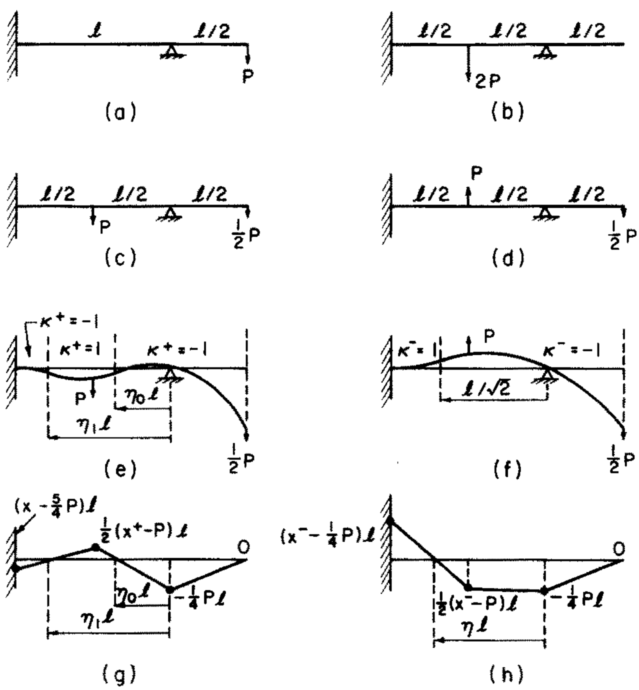

FIG. 1.

are shown in Fig. 1(g–h); the points at which the bending moments vanish are given by

$$\eta_0 = \frac{P}{2(2x^+ - P)}, \qquad \eta_1 = \frac{P}{2(3P - 2x^+)}, \qquad \eta = \frac{3P}{2(2x^- + P)}. \tag{6.2}$$

The actual solution is the one for which the bending moments vanish at the points the curvature changes sign. The solution of (6.1) and (6.2) is then found to be

$$x^+ = 1{\cdot}1858\,P, \qquad \eta_0 = 0{\cdot}3646, \qquad \eta_1 = 0{\cdot}7956,$$
$$x^- = 0{\cdot}5607\,P, \qquad \eta = 0{\cdot}7071.$$

The plastic moment of the beam, formed by superposition of the absolute values of the moments in Fig. 1(g–h), is shown in Fig. 2.

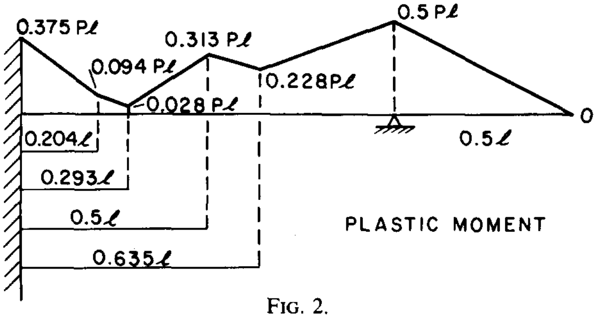

FIG. 2.

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Резюме—Эта работа касается оптимального пластичного проектирования балок, рам и ферм слоистой конструкции для переменных условий нагрузки. Верхние и нижние пределы оптимального веса балки разделяются как для одного, так и для переменных условий нагрузки. Эти теоремы предела используются для создания принципа суперпозиции. Если для участков сечения не предусмотрено определенных пределов, то оптимальное проектирование для переменных условий нагрузки $\mathbf{P}_1$ и $\mathbf{P}_2$ можно получить суперпозицией на одно условие нагрузки $\frac{1}{2}(\mathbf{P}_1 + \mathbf{P}_2)$ и $\frac{1}{2}(\mathbf{P}_1 - \mathbf{P}_2)$. Если разрезы будут иметь, по крайней мере, определенные ненулевые значения, то этот принцип предоставляет верхний и нижний пределы для оптимального веса.

Этот принцип иллюстрируется простым примером.